Kuwait University
Department of Mathematics — Second In-term Test, 8 May 2011
Math 101: Calculus I — Duration 90 minutes

Answer all of the following questions.
Calculators and mobile telephones are not allowed.

1. [3 pts.] Find an equation of the tangent line to the curve $x^3 + 2y^3 = 4xy + 2$ at the point $P(2,1)$.

2. [3 pts.] A point $P(x,y)$ is moving on the curve $y = x^2 - 3$, such that the x-coordinate is increasing at a rate of 7 cm/sec. What is the rate of change of the y-coordinate when the point P reaches $(3,6)$?

3. [3 pts.] Use linear approximation to estimate $\sqrt{16.08}$.

4. [3 pts.] At which points on the curve $y = 10x^3 - 3x^5 + 5$ for $-2 \leqslant x \leqslant 2$ does the tangent line have the greatest slope?

5. [3 pts.] Does there exist a function f such that $f(1) = 1$, $f(3) = 6$, and $f'(x) \leqslant 2$ for all $x \in \mathbb{R}$? Justify your answer.

6. Consider the curve $y = f(x)$ where

$$f(x) = \frac{x}{x^2 - 1}, \quad f'(x) = -\frac{x^2 + 1}{(x^2 - 1)^2}, \quad \text{and} \quad f''(x) = \frac{2x(x^2 + 3)}{(x^2 - 1)^3}.$$

   (a) [2 pts.] Find the horizontal and vertical asymptotes.
   (b) [2 pts.] Find the intervals on which f is increasing and on which f is decreasing, and, determine the local extrema of f.
   (c) [2 pts.] Study the concavity of f and find the points of inflection on the curve.
   (d) [2 pts.] Find the x-intercepts and the y-intercept, and, study the symmetry of the curve.
   (e) [2 pts.] Sketch the graph of the curve.

1. By implicit differentiation,

$$3x^2 + 6y^2\frac{dy}{dx} = 4\left(y + x\frac{dy}{dx}\right).$$

Substitution of $(x, y) = (2, 1)$ implies

$$3(2^2) + 6(1^2)\frac{dy}{dx} = 4\left(1 + 2\frac{dy}{dx}\right).$$

This gives the slope of the tangent line

$$\frac{dy}{dx} = 4.$$

The point-slope formula yields the equation

$$y - 1 = 4(x - 2),$$

which simplifies to

$$y = 4x - 7.$$

2. Since $y = x^2 - 3$, by the chain rule

$$\frac{dy}{dt} = 2x\frac{dx}{dt} \qquad (*)$$

for any variable t. So, when $dx/dt = 7$ and $(x, y) = (3, 6)$,

$$\frac{dy}{dt} = 2(3)7 = 42.$$

Answer: the y-coordinate is increasing at a rate of 42 cm/sec.
(There is a temptation to think of x and y in cm, and t as time in sec. This gives rise to difficulty, because then the units of the left-hand side of $(*)$ are cm/sec while the units of the right-hand side are cm^2/sec. The correct approach is to recognize that x and y are real numbers and therefore dimensionless. Hence, the units of dy/dt on the left-hand side of $(*)$ are the same as those of dx/dt on the right-hand side, whatever t might represent and whatever the units of dx/dt might be.)

3. Let $f(x) = \sqrt{x}$. Then for x close to 16,

$$f(x) \approx f(16) + f'(16)(x - 16) = \sqrt{16} + \frac{1}{2\sqrt{16}}(x - 16) = 4 + \frac{x - 16}{8}.$$

Hence,

$$\sqrt{16.08} \approx 4 + \frac{16.08 - 16}{8} = 4 + \frac{0.08}{8} = 4.01.$$

4. The slope of the tangent line is

$$\frac{dy}{dx} = 30x^2 - 15x^4 = 15x^2(2 - x^2).$$

Therefore, the objective is to find the absolute maxima of $f(x) = 15x^2(2 - x^2)$ for $-2 \leqslant x \leqslant 2$.
The maxima of f are either at critical numbers within the interval or endpoints.

$$f'(x) = 60x - 60x^3 = 60x(1 - x)(1 + x).$$

Hence, the critical numbers of f are -1, 0 and 1.

$$f(\pm 1) = 15(1^2)(2 - 1^2) = 15.$$

$$f(0) = 15(0^2)(2 - 0^2) = 0.$$

$$f(\pm 2) = 15(2^2)(2 - 2^2) = 15(4)(-2) < 0.$$

Therefore, the maxima of f are at -1 and 1.
When $x = -1$ then $y = 10(-1)^3 - 3(-1)^5 + 5 = -2$.
When $x = 1$ then $y = 10(1)^3 - 3(1)^5 + 5 = 12$.
Answer: $(-1, -2)$ and $(1, 12)$.

5. Suppose that there is a function f with the stated properties. Since $f'(x)$ exists for all x, necessarily f is continuous on $[1, 3]$ and differentiable on $(1, 3)$. Consequently, by the Mean Value Theorem, there is a number $c \in (1, 3)$ such that

$$f'(c) = \frac{f(3) - f(1)}{3 - 1} = \frac{6 - 1}{2} = \frac{5}{2} > 2.$$

However, this contradicts the requirement that $f'(x) \leqslant 2$ for all x.
Therefore, such a function f does not exist.

6. (a)

$$\lim_{x \to \pm\infty} f(x) = \lim_{x \to \pm\infty} \frac{1}{x} \frac{1}{1 - \dfrac{1}{x^2}} = 0 \frac{1}{1 - 0} = 0.$$

Hence the curve has exactly one horizontal asymptote, i.e. $y = 0$.
Since f has an infinite discontinuity at -1 and at 1, the curve has exactly two vertical asymptotes, i.e. $x = -1$ and $x = 1$.

(b) Because $x^2 + 1 > 0$ for all x, and $(x^2 - 1)^2 > 0$ for all $x \neq \pm 1$, necessarily $f'(x) < 0$ for all $x \neq \pm 1$.
Answer: There is no interval on which f is increasing, f is decreasing on the intervals $(-\infty, -1)$, $(-1, 1)$, and $(1, \infty)$, and, f has no local extrema.

(c) $f''(x) = 0$ if and only if $x = 0$.

Interval	$(-\infty,-1)$	$(-1,0)$	$(0,1)$	$(1,\infty)$
x	$-$	$-$	$+$	$+$
x^2+3	$+$	$+$	$+$	$+$
$x-1$	$-$	$-$	$-$	$+$
$x+1$	$-$	$+$	$+$	$+$
$f''(x)$	$-$	$+$	$-$	$+$
f	concave downw.	concave upw.	concave downw.	concave upw.

When $x = 0$ then $y = 0$.
Answer: f is concave downward on $(-\infty,-1)$ and $[0,1)$, f is concave upward on $(-1,0]$ and $(1,\infty)$, and the curve has exactly one point of inflection, i.e. at $(0,0)$.

(d) $f(x) = 0$ implies $x = 0$. So there is exactly one x-intercept, i.e. 0.
Because $f(0) = 0$, the y-intercept is 0.

$$f(-x) = \frac{-x}{(-x)^2-1} = -\frac{x}{x^2-1} = -f(x).$$

Hence, f is an odd function, and the curve is symmetric about the origin.

(e)